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Reliability analysis of a new *k*-out-of-*n*: G system with multi-type components and continuous random weights

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Abstract

Traditional weighted *k*-out-of-*n* systems often assume fixed component weights, which limits their applicability in complex engineering scenarios where component weights exhibit continuous randomness. This limitation is particularly critical in redundant safety instrumented systems with common cause failures, such as renewable energy systems, where components experience heterogeneous degradation, dependent failure patterns, and multi-stage performance decline. To address this limitation, this study proposes a novel *k*-out-of-*n*: G system that incorporates multi-type components with continuous random weights, and evaluates its reliability and mean time to failure across two operating states: normal operating state and degraded state. First, the system's operating mechanism is defined, incorporating both the number of surviving components and their sum of weights. Second, the distribution of the system Integrity and the system reliability are derived, considering the dependencies modeled by different Copula functions. Then, Monte Carlo simulations are conducted to investigate the impacts of different dependence structures (fully dependent, inter-class dependent, fully independent) and lifetime correlation patterns on system reliability. Furthermore, the simulation results indicate that mis-specifications of components' relationships can lead to estimation errors. Additionally, the presence of dependence among components can enhance overall system reliability to some extent. Finally, a numerical case study of an aircraft pneumatic system validates the feasibility and practicality of the proposed model.

Keywords: continuous random weight, *k*-out-of-*n* system with multi-type components, degraded state, Copula-based dependence, mean time to failure (MTTF)

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1. Introduction

The k-out-of-n system, as a special case of the weighted k-out-of-n system, has been extensively studied [1–4]. In contrast to the traditional k-out-of-n (denoted as k/n) system (where each component contributes equally to the system performance), the weighted k/n system incorporates weights to reflect the varying contributions of individual components to the system. The weighted k/n: G system functions normally if and only if the total weight (capacity) of the operating components is no less than a predetermined threshold k. Extensive applications of the weighted k/n system can be observed in diverse fields. For example, in power systems [5, 6], communication networks [7], and the aerospace field [8–10], different components exert significantly varying impacts on system performance, which makes them more amenable to weight-based modeling.

Reliability analysis of weighted k/n system composed of multi-type components is a prevalent issue in current engineering fields. In wind turbines, for instance, gearboxes and lubrication systems exert distinct impacts on the overall system performance, with differing failure modes and degradation rates. Similarly, data center server clusters typically encompass both compute nodes and storage nodes, which vary in terms of reliability requirements and maintenance strategies. Treating all components as a single type would fail to accurately reflect their actual contributions. To capture the diverse functional characteristics, performance levels, and degradation patterns of components in real-world engineering systems, modeling weighted k/n systems with multi-type components holds significant theoretical and practical value.

The system featuring independent multi-type components has been thoroughly explored through diverse perspectives. In the case of independent components, Hamdan *et al* [11] proposed a weighted k/n system with multi-type independent components, that utilizing survival signatures to simplify the reliability computation of the systems. Zhang [12] studied the reliability of heterogeneous independent random-weighted k/n systems and performed stochastic comparisons of selection strategies under the usual stochastic order. Mahmoudi and Meshkat [13] put forward a specific case of the weighted k/n system, comprising two different types of components, where each has its own weight and reliability. Franko *et al* [14] examined the impact of a cold storage component on the reliability of a weighted k/n system, which comprises two distinct types of mutually independent components. Dembińska and Jasiński [15] conducted parametric inference for k/n systems with independent components that follow different lifetime distributions, using data collected at the time of system failure. Jasiński [16] focused on coherent systems composed of different types of components, aiming to study the number of failed components of each type when the system fails, and apply the results to the classical age replacement policy to determine the optimal replacement time. When components are mutually independent and their lifetimes follow exponential distributions, Eryilmaz and Ucum [17] derived the distribution of the lost capacity upon system failure, and obtained the optimal number of standby components under different costs by solving an optimization problem. Che *et al* [18] employed the universal generating function (UGF) method to evaluate the reliability of a multi-state weighted k/n system composed of n man-machine units (with independence among units and dependence within each unit), and calculated the probability that the total system performance meets the threshold k. As a special case, the weighted k/n system composed of components with identical lifetime distributions have been extensively studied [19, 20]. Complex systems typically exhibit significant correlations among their components' life, and Copula functions are commonly employed to model such correlations. For instance, Eryilmaz [21] first applied Copula theory to the reliability and mean time to failure (MTTF) analysis of dynamic weighted k/n system with dependent components. Salehi *et al* [22] extended Eryilmaz model [21] to the weighted k/n system with a randomly selected number of components, finding that the randomness of the component selection significantly affects the system reliability. Mahmoudi *et al* [23] further extended the weighted k/n system in two aspects: from two types of components to multiple types, and from a single Copula to mixed Copula. For more articles that analyze the system reliability in the context of dependent component lifetimes, one may refer to Roy and Gupta [24], Fang *et al* [25], Ozkut [26], Andersen *et al* [27].

The weight of components can represent their ability to accomplish or undertake specified functions [28–31], which are typically modeled as either single-state fixed or multi-state fixed form. The single-state fixed form indicates that the weight of component remains a constant throughout the entire system operation process, which has been extensively studied [28, 29, 32]. In recent years, as reliability problems have grown more complex, the multi-state fixed form of weight has attracted attention from researchers. The multi-state fixed weight form means that the component's weight is a different constant for each different state. Larsen *et al* [30] proposed a weighted k/n system featuring multi-state components with fixed form weights and evaluated its reliability via the UGF method. Zhuang *et al* [31] proposed a recursive method to evaluate system reliability based on Larsen *et al*'s model [30]. Eryilmaz and Bozbulut [33] established sufficient conditions to compare the expected total weights in weighted k/n system with components having three distinct states: perfect functioning, partial working, and complete failure. Li *et al* [34] proposed a weighted k/n system with N multi-state components. These components have predetermined discrete states from 0 to M, where M denotes the perfect functioning state and 0 indicates complete failure. Bisht and Singh [35] developed an Lz-transform-based algorithm to compute reliability of multi-state weighted k/n system in dynamic environments.

Departing from the fixed-weight assumption, Eryilmaz [36] analyzed the k/n system with random component weights, assigning each component an integer-valued weight uniformly sampled from a specified range of positive values. Sun *et al* [37] investigated optimal allocation strategies for hot standby components in weighted k/n system under different weight distributions of components, including both continuous and discrete ones. Lorvand and Zarezadeh [38] established a shock reliability model for a weighted k/n system, where the component's weight decreases monotonically with the number of the random shocks. The aforementioned literature demonstrates the random characteristics of weights in various senses. However, in practical engineering, component performance degradation is a continuous random process, and the accompanying weights also exhibit dynamic characteristics of continuous random decline, showing different weight reduction processes with variations in component performance. For instance, the computing power of each server degrades over time due to reduced heat dissipation efficiency or hardware aging; the thrust output of aero-engines gradually decreases as the service life increases. These examples indicate that weight loss is a continuous and random dynamic process. To more accurately predict the system reliability at any time, a dynamic stochastic weight model is introduced in this paper.

In an existing weighted k/n system, the failure threshold is determined by whether the total component weight meets a predefined critical value, while ignoring the number of working parts [28, 39–41]. However, practical engineering applications like nuclear power plant cooling systems show that reliable operation requires the simultaneous satisfaction of two conditions: (1) The number of normally operating pumps exceeds the minimum requirement; (2) The total flow rate (weight) of the remaining pumps is sufficient to sustain the system's safe operation. In the unmanned aerial vehicle (UAV) formation communication network, mission failure occurs if the number of surviving UAVs falls below the threshold or the total communication bandwidth (weight) is insufficient. These two failure criteria are more aligned with the requirements of safety-critical systems than the single-weight failure criteria. This paper proposes a dual failure mode that integrates both *the number of surviving components* and *the total weight*.

Existing studies on weighted k/n systems typically assume that discrete component weight loss, while neglecting the interaction effects between a system's degradation process and its state. In practice, The gradual increase in a system's degradation quantity leads to a corresponding gradual decline in its capacity to accomplish tasks. Meanwhile, when damage reaches a critical threshold, the system enters a phase of qualitative change, where weight loss accelerates gradually and continuously. For instance, in data center server clusters, cooling system failures induce elevated ambient temperatures, causing accelerated CPU performance throttling due to thermal constraints.

To address the limitations in existing research, such as discretized weight modeling, oversimplified system failure modes, insufficient sensitivity analysis of component interdependencies, and the absence of bidirectional interaction mechanisms between the system and its components. The main innovations of this research are summarized as follows:

- This study proposes a dynamic random weight model that overcomes the traditional approach of treating weights as fixed values or discrete changes [36, 42]. Based on random process theory, component weight degradation is modeled as a random process, capturing the realistic evolution of performance decline in practical engineering scenarios. This approach provides a more refined dynamic assessment method for system reliability.
- This study constructs a bidirectional interaction mechanism between system states and component degradation, overcoming the traditional unidirectional influence assumption. The model establishes that system state transitions (normal operating state, degraded state and failure state) directly accelerate component weight degradation rates. This captures the cascading degradation phenomenon commonly observed in engineering practice and provides more reliable theoretical support for preventive maintenance strategies.
- This study incorporates multiple component dependency structures into the reliability analysis framework, going beyond traditional single-assumption approaches. Three typical dependency modes are systematically investigated: fully dependent, intra-class dependent (inter-class independent), and completely independent. By comparing system reliability under different dependency structures, this approach identifies key patterns affecting system robustness and reveals the influence mechanisms of component dependencies on system performance.

The subsequent content of this article is structured as follows: section 2 elaborates the characteristics of the model studied in this paper; section 3 models the reliability and completeness of the dynamic weighted k/n system; section 4 provides the system reliability and completeness under special cases; section 5, the numerical simulation section, employs Monte Carlo to simulate the system's reliability, completeness and MTTF, and verifies the model's validity through numerical examples; section 6, the conclusion section, summarizes the key finding of this paper.

2. Model assumptions and definitions

2.1. Model assumptions

Consider a system composed of n components, which are categorized into m mutually exclusive categories $S_1, S_2, \ldots, S_m$ (where $|S_1| = n_1, |S_2| = n_2, \ldots, |S_m| = n_m$, and $\sum_{k=1}^{m} n_k = n$). The proposed two-stage continuous weighted k/n: G system with multi-type components (denoted briefly as the $(w_t - k)_2/n$ system) exhibits the following characteristics for components and the system, respectively:

(I) The components of the system exhibit the following features:
  (a) Each component has two states: operating state and failure state. Its state is solely determined by the lifetime distribution.
  (b) Let X_{ij} denote the lifetime of the jth component in category i, where $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n_i$. Components within the same category i share the same lifetime cumulative distribution function (CDF)

$F_i(x)$ and the same initial weight $W_{ij}(0) = W_i(0)$. The weight of the jth component in category i at time t is given by $W_{ij}(t)$. Where $W_{ij}(t)$ depends on the component's performance degradation.
(c) $W_{ij}(t)$ denotes the weight of the jth component in category i at time t, which depends on the initial weight $W_i(0)$ and the incremental weight attenuation.
(d) The components are non-repairable, and their lifetimes exhibit dependence characterized by a Copula function.

(II) The system exhibits the following features:
(a) The system operates in one of three states: normal operating state, degraded state , and failure state.
(b) At time t, the system weight equals the sum of the weights of surviving components.
(c) The system is in the normal operating state if at least K_1 components function properly and their total weight is no less than W_1. If the above normal operating conditions are not met, but at least K_2 components work properly with a total weight of no less than W_2, the system is in the degraded state.
(d) The system enters the failure state if the number of working components drops below K_2, or the total weight of working components falls below W_2. Here, K_1, K_2, W_1, W_2 are fixed constants satisfying $K_2 < K_1$ and $W_2 < W_1$.
(e) When the system enters the degraded state, component weights $W_{ij}(t)$ decline more rapidly, exhibiting a two-stage degradation pattern driven by system state transitions.

Let $K(t)$ denote the number of surviving components at time t, and $W(t)$ represent the system weight at time t. Based on the model assumption in (II), we define the sets Ω, Ω_1, Ω_2, and Ω_3 as follows:

$$\Omega = \{t : t \geqslant 0\}$$

$$\Omega_1 = \{t : K(t) \geqslant K_1, W(t) \geqslant W_1\}$$

$$\Omega_2 = \{t : K(t) < K_1 \text{or } W(t) < W_1\}$$

$$\Omega_3 = \{t : K(t) < K_2 \text{or } W(t) < W_2\}$$

The system state $\psi(t)$ at time t is defined as

$$\psi(t) = \begin{cases} \text{normal operating state}, & t \in \Omega_1, \\ \text{degraded state}, & t \in \Omega_2 - \Omega_3, \\ \text{failure state}, & t \in \Omega_3. \end{cases}$$

The system state at time t is illustrated in figure 1, where point A represents the initial perfect state of the system at $t = 0$.

The distinct colored areas in figure 1 represent the dynamic evolution process of the system's operating states: the green area corresponds to the normal operating state; the yellow region indicates the degraded state, which the system enters as component performance deteriorates during operation; and the red region represents the failure state, triggered by further deterioration of component performance.

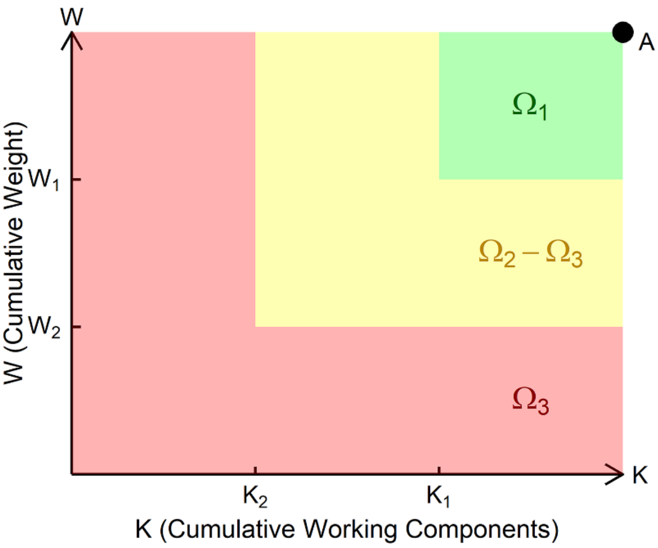

Figure 1. System state.

2.2. Definitions

Definition 1 (System normal operating time). The system normal operating time T_1 is defined as the first hitting time when the system can no longer maintain normal operating state:

$$T_1 = \min\{T_{1a}, T_{1b}\} \tag{1}$$

where $T_{1a} = \inf\{t : K(t) < K_1\}$; $T_{1b} = \inf\{t : W^{(1)}(t) < W_1\}$; $K(t) = \sum_{i=1}^{m}\sum_{j=1}^{n_i} I\{X_{ij} > t\}$ is the number of surviving components; $W^{(1)}(t) = \sum_{i=1}^{m}\sum_{j=1}^{n_i}[W_i(0) - \Delta W_{ij}^{(1)}(t)]I\{X_{ij} > t\}$ is the total weight under normal operating state. $\Delta W_{ij}^{(1)}(t)$ is the cumulative weight loss governed by the degradation parameters in the normal operating state.

Definition 2 (System operating time). The system operating time T_2 is defined as the first hitting time when the system fails:

$$T_2 = \min\{T_{2a}, T_{2b}\} \tag{2}$$

where $T_{2a} = \inf\{t : K(t) < K_2\}$; $T_{2b} = \inf\{t : W^{(2)}(t) < W_2\}$; $K(t) = \sum_{i=1}^{m}\sum_{j=1}^{n_i} I\{X_{ij} > t\}$ is the number of surviving components; $W^{(2)}(t) = \sum_{i=1}^{m}\sum_{j=1}^{n_i}[W^{(1)}(T_1) - \Delta W_{ij}^{(2)}(t - T_1)]I\{X_{ij} > t\}$ is the total weight under degraded state. $\Delta W_{ij}^{(2)}(t)$ is the cumulative weight loss governed by degraded-state degradation parameters.

According to model assumption (II,e), the component weight deterioration process follows a two-stage random process with change point T_1. Distinctly, $\Delta T = T_2 - T_1$ is the system degraded state operating time.

Definition 3 (Component weight). Let $T_1 = t_1$ is the change point of system operation state from normal to degraded, the weight $W_{ij}(t)$ of the jth component in category i is:

$$W_{ij}(t) = \begin{cases} W_i(0) - \Delta W_{ij}^{(1)}(t), & t \leqslant t_1 \\ W_i(0) - \Delta W_{ij}^{(1)}(t_1) - \Delta W_{ij}^{(2)}(t - t_1), & t > t_1. \end{cases} \quad (3)$$

According to definition 3, the total system weight $W(t)$ of the weighted k/n system are given by

$$W(t) = \sum_{i=1}^{m} \sum_{j=1}^{n_i} W_{ij}(t) I_{ij}(t) \quad (4)$$

where $I_{ij}(t)$ denotes the indicator function:

$$I_{ij}(t) = \begin{cases} 1, & X_{ij} \geqslant t, \\ 0, & X_{ij} < t. \end{cases}$$

Measuring the system's operational probability in each state is crucial for formulating effective maintenance strategies. Therefore, we introduce the following definition:

Definition 4 (System integrity function). The system integrity function $G(t)$ is defined as the survival function of the system normal operating time T_1:

$$G(t) = P\{T_1 > t\} = P\left\{K(t) \geqslant K_1 \text{and } W^{(1)}(t) \geqslant W_1\right\}, \quad (5)$$

where $T_1 = \min\{T_{1a}, T_{1b}\}$ is defined in definition 1.

$G(t)$ quantifies the probability that the system maintains its normal operating state at time t, which represents the system's capability to deliver optimal performance. The probability density function of T_1 is $f_{T_1}(t) = -\frac{\mathrm{d}G(t)}{\mathrm{d}t}$. Obviously, $G(0) = 1$ and $G(t)$ is non-increasing.

Based on the model established in section 2.1 and definition 2, the system reliability function $R(t)$ is defined as follows:

Definition 5 (System reliability function). The system reliability function $R(t)$ is defined as the survival function of the system operating time T_2:

$$R(t) = P\{T_2 > t\} = P\{K(t) \geqslant K_2 \text{ and } W(t) \geqslant W_2\}, \quad (6)$$

where $T_2 = \min\{T_{2a}, T_{2b}\}$ is defined in definition 2.

$R(t)$ quantifies the probability that the system remains operational (including normal and degraded states) at time t, thereby characterizing its ability to deliver minimally acceptable functionality before complete failure and providing a crucial metric for reliability and safety evaluation. The relationship between $G(t)$ and $R(t)$ is given by theorem 1.

From a definitional perspective, system integrity essentially satisfies the definition of generalized reliability. However, it is termed 'integrity' and defined with more stringent requirements than reliability in this study based on two primary considerations: first, it provides quantifiable state thresholds for precise maintenance timing. Second, it is critical for high-performance systems(e.g. data centers, aircraft engines) where degradation has severe consequences, making integrity a more operationally relevant metric than conventional reliability. This distinction is mathematically formalized through definitions 4 and 5, and their relationship is rigorously established in theorem 1.

Theorem 1. *For all $t \geqslant 0$, the system integrity is always less than or equal to the system reliability, i.e. $G(t) \leqslant R(t)$.*

Proof: See appendix A. $R(t) - G(t)$ is the probability of degraded operation, representing the duration for which the system can continue functioning in a degraded state. It reflects the system's ability to operate with reduced performance before complete failure and can be used to evaluate pre-failure degradation behavior, helping to develop appropriate maintenance strategies and improve overall system reliability. □

3. Copula-based modeling of system integrity and reliability

Using Copula method to characterize the dependence among heterogeneous components, this section analyzes the reliability of the $(w_t - k)_2/n$ system by deriving both the system integrity function $G(t)$ and the system reliability function $R(t)$.

3.1. Copula theory

An n-dimensional Copula function is a joint distribution function of standard uniform marginal distributions, and it has the following three properties [43]:

- $C(\mathbf{u}) = C(u_1, u_2, \ldots, u_N)$ increases with u_i;
- $u_i \in [0,1]$ for all $i \in \{1, \ldots, N\}$;
- For all $u_{1,i_1}, \ldots, u_{N,i_N} \in [0,1]^N$, $i_1, \ldots, i_N = \{1,2\}$ with $u_{j,1} \leqslant u_{j,2}$ for any
- $j \in \{1,2,\ldots,N\}$, one has $\sum_{i_1=1}^{2} \cdots \sum_{i_N=1}^{2} (-1)^{i_1+\cdots+i_N} C(u_{j,i_1}, \ldots, u_{j,i_N}) \geqslant 0$.

Sklar's theorem shows that if all real-valued random variables $\{T_1, T_2, \ldots, T_n\}$ comply with a set of marginal distributions $\{F_1(t), F_2(t), \ldots, F_n(t)\}$ and a joint distribution function $H(\cdot)$, there exists a Copula function $C(\cdot)$ conforming to the relation:

$$H(T_1, T_2, \ldots, T_n) = C(F_1(t), F_2(t), \ldots, F_n(t)). \quad (7)$$

3.2. On the properties and parameter estimation of Clayton and Gumbel copulas

The selection of an appropriate copula family should be guided by the physical failure mechanisms and empirical dependency

Table 1. Summary of properties of Clayton and Gumbel Archimedean Copulas.

Copula	Distribution function $C(u_1,\ldots,u_n;\theta)$	Parameter(θ)	Tail dependence coefficient	Tail dependence
Clayton	$\left(\sum_{i=1}^{n} u_i^{-\theta}-n+1\right)^{-1/\theta}$	$\theta\in[-1,\infty)\setminus\{0\}$	$\lambda_U=0,\lambda_L=2^{-1/\theta}$	Lower tail
Gumbel	$\exp\left\{-\left[\sum_{i=1}^{n}(-\ln u_i)^{\theta}\right]^{1/\theta}\right\}$	$\theta\in[1,\infty)$	$\lambda_U=2-2^{1/\theta},\lambda_L=0$	Upper tail
The relationship between Kendall's τ and copula parameter				
Clayton	$\tau_n(\theta)=\frac{1}{2^{n-1}-1}\cdot 2^n\prod_{m=0}^{n-1}\left(\frac{1+m\theta}{2+m\theta}\right)-1$			
Gumbel	$\tau_n(\theta)=\frac{1}{2^{n-1}-1}\left[2^{n-1}\sum C_{m_1\ldots m_n}\frac{(m-1)!}{(n-1)!}\left(\frac{1}{2}\right)^{\theta(m-1)}\prod_{i=1}^{n}\frac{\Gamma(i-\frac{1}{\theta})}{\Gamma(1-\frac{1}{\theta})}-1\right]$			

patterns observed in the system. In reliability applications, component lifetimes typically exhibit positive dependence [23]. The Clayton and Gumbel copulas, as two important members of the Archimedean copula family, are particularly well suited for reliability modeling because they can capture different patterns of positive tail dependence. The Gumbel copula characterizes the probability that multiple components fail simultaneously during extreme stress conditions; conversely, the Clayton copula characterizes the probability that components fail together early in their lifetimes. Table 1 summarizes the key properties of the two copulas, and they are further utilized in the numerical simulations presented later in this paper.

Given N observed system realizations, Kendall's τ for n-dimensional random variables $X_1,\ldots,X_n$ can be estimated using [44]:

$$\hat{\tau}_n=\frac{1}{2^{n-1}-1}\left[\frac{2^n}{n(n-1)}\sum_{i\neq j}I(X_i\leqslant X_j)-1\right] \tag{8}$$

where $I(\cdot)$ is the indicator function. This estimator measures the concordance probability among all component pairs. After obtaining $\hat{\tau}_n$, the copula parameter θ can be estimated by inverting the theoretical relationship between τ_n and θ. For the Archimedean copulas considered in this study, these relationships have been established in the literature [44] and are summarized in table 1. This estimation approach has been successfully applied in reliability analysis by [45].

In engineering, the copula family should be selected through a systematic four-step process. First, physical mechanism analysis identifies dominant failure modes (wear-out vs. random shock vs. infant mortality), assesses common cause failure potential, evaluates environmental stress patterns, and analyzes component interaction mechanisms; for stress-strength systems, the Clayton copula provides a theoretically grounded choice [21]. Second, empirical data examination constructs scatter plots of component lifetime ranks and calculates empirical tail dependence coefficients, along with Kendall's τ as correlation measures. Third, engineering validation verifies that the selected copula's tail behavior matches observed failure clustering patterns and that predicted joint failure probabilities align with field experience.

3.3. Copula-based modeling of system integrity

Property 1. The integrity $G(t)$ of the $(w_t-k)_2/n$ system at time t is derived as follows:

$$G(t)=\underset{a_1+a_2+\cdots+a_m\geqslant K_1}{\sum_{a_1=0}^{n_1}\sum_{a_2=0}^{n_2}\cdots\sum_{a_m=0}^{n_m}} P\left\{\sum_{i=1}^{m}\sum_{j=1}^{n_i}W_{ij}^{(1)}(t)I_{ij}(t)\geqslant W_1\right\}\times\prod_{i=1}^{m}\binom{n_i}{a_i}\left(\sum_{k_1=0}^{a_1}\sum_{k_2=0}^{a_2}\cdots\sum_{k_m=0}^{a_m}(-1)^{k_1+k_2+\cdots+k_m}\prod_{i=1}^{m}\binom{a_i}{k_i}A_{1:m}(t)\right) \tag{9}$$

where a_i denote the number of surviving components in category i at time t, $A_{1:m}(t)=C\big(\underbrace{F_1(t),\cdots,F_1(t)}_{n_1-a_1+k_1},\underbrace{F_2(t),\cdots,F_2(t)}_{n_2-a_2+k_2},\cdots,\underbrace{F_m(t),\cdots,F_m(t)}_{n_m-a_m+k_m};\theta\big)$, and $C(\cdot)$ is a Copula function.

Proof: See appendix B. The probability $P\{\sum_{i=1}^{m}\sum_{j=1}^{n_i}W_{ij}^{(1)}(t)I_{ij}(t)\geqslant W_1\}$ in equation (9) involves the distribution of a sum of random variables. Since no closed-form solution exists, this probability can be evaluated via numerical methods such as convolution or Monte Carlo simulation. If the weight loss increments $\Delta W_{ij}^{(1)}(t)$ are additive, this probability can be further simplified as follows. □

Substituting $W_{ij}^{(1)}(t)=W_i(0)-\Delta W_{ij}^{(1)}(t)I_{ij}(t)$ into the probability expression, where $W_i(0)$ is the initial weight of each component of type i, we have:

$$P\left\{\sum_{i=1}^{m}\sum_{j=1}^{n_i}W_{ij}^{(1)}(t)I_{ij}(t)\geqslant W_1\right\}=P\left\{\sum_{i=1}^{m}a_iW_i(0)-\sum_{i=1}^{m}\sum_{j=1}^{n_i}\Delta W_{ij}^{(1)}(t)I_{ij}(t)\geqslant W_1\right\}.$$

Rearranging the inequality and using the additivity of the gamma process, we obtain:

$$F_{\sum_{i=1}^{m}\sum_{j=1}^{n_i}\Delta W_{ij}^{(1)}(t)I_{ij}(t)}\left(\sum_{i=1}^{m}a_iW_i(0)-W_1\right). \tag{10}$$

3.4. Copula-based modeling of system reliability

Property 2. The reliability of the $(w_t-k)_2/n$ system at time t is derived as follows:

$$R(t)=\int_0^t\left[\sum_{b_1=0}^{n_1}\underset{b_1+b_2+\cdots+b_m\geqslant K_2}{\sum_{b_2=0}^{n_2}}\cdots\sum_{b_m=0}^{n_m}P\left\{\sum_{i=1}^{m}\sum_{j=1}^{n_i}W_{ij}^{(2)}(t|z)\,I_{ij}(t)\geqslant W_2\right\}\times\prod_{i=1}^{m}\binom{n_i}{b_i}\left(\sum_{k_1=0}^{b_1}\sum_{k_2=0}^{b_2}\cdots\sum_{k_m=0}^{b_m}(-1)^{k_1+k_2+\cdots+k_m}\prod_{i=1}^{m}\binom{b_i}{k_i}B_{1:m}(t)\right)\right]\times f_{T_1}(z)\,\mathrm{d}z \tag{11}$$

where, $B_{1:m}(t)=C\Big(\underbrace{F_1(t),\cdots,F_1(t)}_{n_1-b_1+k_1},\underbrace{F_2(t),\cdots,F_2(t)}_{n_2-b_2+k_2},\cdots,\underbrace{F_m(t),\cdots,F_m(t)}_{n_m-b_m+k_m};\theta\Big)$.

Proof: See appendix C. Similarly, if the weight loss increments $\Delta W_{ij}^{(2)}(t)$ satisfy the additivity property, the probability $P\{\sum_{i=1}^{m}\sum_{j=1}^{n_i}W_{ij}^{(2)}(t|z)\ \ I_{ij}(t)\geqslant W_2\}$ can be further derived as:

$$F_{\sum_{i=1}^{m}\sum_{j=1}^{n_i}\left(\left(\Delta W_{ij}^{(2)}(t-z)+\Delta W_{ij}^{(1)}(z)\right)I_{ij}(t)\right)}\left(\sum_{i=1}^{m}b_iW_i(0)-W_2\right). \tag{12}$$

□

3.5. Temporal performance metrics

The temporal metrics associated with system integrity and reliability provide complementary perspectives on system performance:

Mean time to defect (MTTD): The average time from system startup until the system state transitions from normal operating state to degraded state. MTTD is the expected value of T_1:

$$\mathrm{MTTD}=\int_0^{\infty}G(t)\,\mathrm{d}t. \tag{13}$$

Based on property 1, the MTTD of the $(w_t-k)_2/n$ system is given by

$$\mathrm{MTTD}=\int_0^{\infty}\left[\sum_{a_1=0}^{n_1}\underset{a_1+a_2+\cdots+a_m\geqslant K_1}{\sum_{a_2=0}^{n_2}}\cdots\sum_{a_m=0}^{n_m}P\left\{\sum_{i=1}^{m}\sum_{j=1}^{n_i}W_{ij}^{(1)}(t)\,I_{ij}(t)\geqslant W_1\right\}\times\prod_{i=1}^{m}\binom{n_i}{a_i}\left(\sum_{k_1=0}^{a_1}\sum_{k_2=0}^{a_2}\cdots\sum_{k_m=0}^{a_m}(-1)^{k_1+k_2+\cdots+k_m}\prod_{i=1}^{m}\binom{a_i}{k_i}A_{1:m}(t)\right)\right]\mathrm{d}t. \tag{14}$$

Mean time to failure (MTTF): The expected total operating time before complete system failure. MTTF is the expected value of T_2:

$$\mathrm{MTTF}=\int_0^{\infty}R(t)\,\mathrm{d}t$$
$$=\int_0^{\infty}\int_0^t\left[\sum_{b_1=0}^{n_1}\underset{b_1+b_2+\cdots+b_m\geqslant K_2}{\sum_{b_2=0}^{n_2}}\cdots\sum_{b_m=0}^{n_m}P\left\{\sum_{i=1}^{m}\sum_{j=1}^{n_i}W_{ij}^{(2)}(t|z)\,I_{ij}(t)\geqslant W_2\right\}\times\prod_{i=1}^{m}\binom{n_i}{b_i}\left(\sum_{k_1=0}^{b_1}\sum_{k_2=0}^{b_2}\cdots\sum_{k_m=0}^{b_m}(-1)^{k_1+k_2+\cdots+k_m}\prod_{i=1}^{m}\binom{b_i}{k_i}B_{1:m}(t)\right)\right]\times f_{T_1}(z)\,\mathrm{d}z\mathrm{d}t. \tag{15}$$

Mean degraded operation time (MDOT): The expected time spent in the degraded state is given by

$$\mathrm{MDOT}=\mathrm{MTTF}-\mathrm{MTTD}=E[T_2-T_1]=\int_0^{\infty}[R(t)-G(t)]\,\mathrm{d}t. \tag{16}$$

This metric quantifies the expected operational duration in the degraded state, which is critical for optimizing preventive maintenance schedules and determining intervention timing.

4. Special case

The following discussion focuses on the integrity and reliability of the $(w_t-k)_2/n$ system under several special circumstances.

4.1. Special case about component

When state transitions depend only on the total weight of surviving components, properties 1 and 2 reduce to the following simplified forms, respectively:

$$G(t)=\sum_{a_1=0}^{n_1}\sum_{a_2=0}^{n_2}\cdots\sum_{a_m=0}^{n_m}P\Big\{\sum_{i=1}^{m}\sum_{j=1}^{n_i}W_{ij}^{(1)}(t)\,I_{ij}(t)\geqslant W_1\Big\}\times\prod_{i=1}^{m}\binom{n_i}{a_i}\left(\sum_{k_1=0}^{a_1}\sum_{k_2=0}^{a_2}\cdots\sum_{k_m=0}^{a_m}(-1)^{k_1+k_2+\cdots+k_m}\prod_{i=1}^{m}\binom{a_i}{k_i}A_{1:m}(t)\right) \tag{17}$$

$$R(t)=\int_0^t\left[\sum_{b_1=0}^{n_1}\sum_{b_2=0}^{n_2}\cdots\sum_{b_m=0}^{n_m}P\Big\{\sum_{i=1}^{m}\sum_{j=1}^{n_i}W_{ij}^{(2)}(t|z)\,I_{ij}(t)\geqslant W_2\Big\}\times\prod_{i=1}^{m}\binom{n_i}{b_i}\left(\sum_{k_1=0}^{b_1}\sum_{k_2=0}^{b_2}\cdots\sum_{k_m=0}^{b_m}(-1)^{k_1+k_2+\cdots+k_m}\prod_{i=1}^{m}\binom{b_i}{k_i}B_{1:m}(t)\right)\right]\times f_{T_1}(z)\,\mathrm{d}z. \tag{18}$$

When components fail independently, $G(t)$ and $R(t)$ admit further simplifications:

$$G(t)=\sum_{a_1=0}^{n_1}\sum_{a_2=0}^{n_2}\cdots\sum_{a_m=0}^{n_m}P\left\{\sum_{i=1}^{m}\sum_{j=1}^{n_i}W_{ij}^{(1)}(t)I_{ij}(t)\geqslant W_1\right\}\times\prod_{i=1}^{m}\binom{n_i}{a_i}[F_i(t)]^{n_i-a_i}[1-F_i(t)]^{a_i} \tag{19}$$

$$R(t)=\int_0^t\left[\sum_{b_1=0}^{n_1}\sum_{b_2=0}^{n_2}\cdots\sum_{b_m=0}^{n_m}P\left\{\sum_{i=1}^{m}\sum_{j=1}^{n_i}W_{ij}^{(2)}(t|z)I_{ij}(t)\geqslant W_2\right\}\times\prod_{i=1}^{m}\binom{n_i}{a_i}[F_i(t)]^{n_i-b_i}[1-F_i(t)]^{b_i}\right]f_{T_1}(z)\,\mathrm{d}z. \tag{20}$$

4.2. Special case about weight

Typically, random process is used to model component performance degradation, such as crack propagation. As discussed in section 1, component weight degradation is a manifestation of performance degradation. Therefore, weight degradation over time can be modeled as a monotonically decreasing stochastic process, such as the Gamma process or Inverse Gaussian process.

In this paper, we assume that the weight degradation increment $\Delta W_i^{(1)}(t)$ in the normal operating state at time t follows a Gamma distribution $\mathrm{Ga}(\alpha_i^{(1)}t,\beta)$, while the weight degradation increment $\Delta W_i^{(2)}(t-z)$ in the degraded state follows $\mathrm{Ga}(\alpha_i^{(2)}(t-z),\beta)$. Consequently, the system integrity $G(t)$ and reliability $R(t)$ can be expressed as follows:

$$G(t)=\sum_{a_1=0}^{n_1}\sum_{\substack{a_2=0\\a_1+a_2+\cdots+a_m\geqslant K_1}}^{n_2}\cdots\sum_{a_m=0}^{n_m}\left[\frac{\gamma\left(\sum_{i=1}^{m}a_i\alpha_i^{(1)}t,\beta\left(\sum_{i=1}^{m}a_iW_i(0)-W_1\right)\right)}{\Gamma\left(\sum_{i=1}^{m}a_i\alpha_i^{(1)}t\right)}\times\prod_{i=1}^{m}\binom{n_i}{a_i}[F_i(t)]^{n_i-a_i}[1-F_i(t)]^{a_i}\right] \tag{21}$$

$$R(t)=\int_0^t\left[\sum_{b_1=0}^{n_1}\sum_{\substack{b_2=0\\b_1+b_2+\cdots+b_m\geqslant K_2}}^{n_2}\cdots\sum_{b_m=0}^{n_m}\times\left[\frac{\gamma\left(\sum_{i=1}^{m}b_i\alpha_i^{(1)}z+\sum_{i=1}^{m}b_i\alpha_i^{(2)}(t-z),\beta\left(\sum_{i=1}^{m}b_iW_i(0)-W_2\right)\right)}{\Gamma\left(\sum_{i=1}^{m}b_i\alpha_i^{(1)}z+\sum_{i=1}^{m}b_i\alpha_i^{(2)}(t-z)\right)}\times\prod_{i=1}^{m}\binom{n_i}{a_i}[F_i(t)]^{n_i-a_i}[1-F_i(t)]^{a_i}\right]\right]f_{T_1}(z)\,\mathrm{d}z \tag{22}$$

where $\Gamma(\cdot)$ is gamma function, and $\gamma(\cdot)$ is lower incomplete gamma function.

5. Data simulation

Four sets of Monte Carlo experiments were designed to evaluate the impact of weight degradation processes, component dependence, and the choice of Copula function on the reliability of the proposed system. Following Eryilmaz's framework [21], we examine three dependence levels (Kendall's $\tau=0.29,0.68,0.81$), corresponding to weak, moderate, and strong dependence. These values, chosen to reflect empirical ranges in engineering reliability studies [23], are used to calibrate the Copula parameters for the analysis. Regarding the lifetime distribution of a component, the Weibull distribution, denoted as Weibull (a,b), is widely used in reliability engineering due to its flexibility in describing various failure modes, serving as a key tool for analyzing product lifetimes, evaluating reliability metrics, and performing lifetime prediction. By fitting historical failure data, the shape a and scale b can be estimated by Maximum Likelihood Estimation [46]. We adopt the exponential and Weibull distributions to model the lifetimes of the component types, with parameters assumed to be known from historical data.

Table 2. Simulation parameters.

Parameter name	Type A	Type B
Number	$n_1=3$	$n_2=2$
Distribution	Exp(0.04)	Weibull(2, 28)
Initial weight	$W_1(0)=14$	$W_2(0)=15$
Stage-1 weighted loss function	$\mathrm{Ga}(9t,0.01)$	$\mathrm{Ga}(8t,0.01)$
Stage-2 weighted loss function	$\mathrm{Ga}(11t,0.02)$	$\mathrm{Ga}(10t,0.02)$
System state	Normal	Degraded
Threshold symbol	K_1 W_1	K_2 W_2
Threshold value	5 53	3 24

Assume that the $(w_t-k)_2/n$ system contains two types of components, i.e. Type A and Type B. The weight degradation increments for both types follow Gamma distributions. All specific parameters are presented in table 2, with other model assumptions consistent with those established in section 2.

5.1. Simulation and comparative analysis

Experiment 1. Weight degradation processes and parameter uncertainty analysis.

Under the parameter configuration in table 2, we consider the case where all components are correlated via a Gumbel Copula with $\tau=0.29$. The simulated weight degradation processes for the system and its components are shown in figure 2. It can be observed that the system transitions to the degraded state at the two-stage change point when the number of functioning components falls below K_1, and ultimately fails when the total weight of surviving components reaches the system's weight-based failure threshold W_2. Moreover, the total system weight drops sharply whenever a component fails. After the system transitions from the normal operating state to the degraded state, each component's weight degradation exhibits a more pronounced attenuation trend compared to that in the previous state.

The simulation results of system integrity and reliability are presented in figure 3, with specific time-point values provided in table 3. As shown in figure 3, the integrity curve $G(t)$ remains consistently below the reliability curve $R(t)$ across the

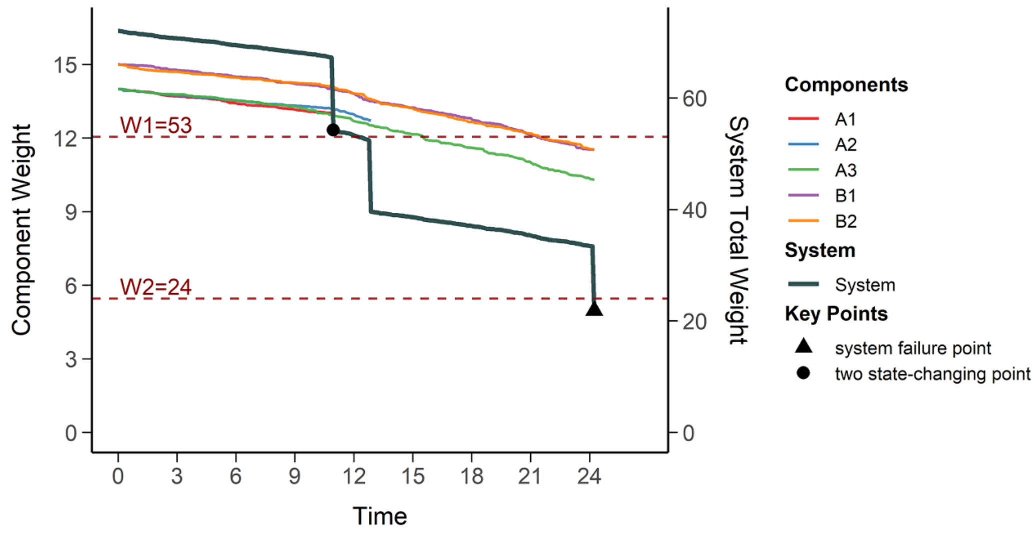

Figure 2. System and component weight degradation processes.

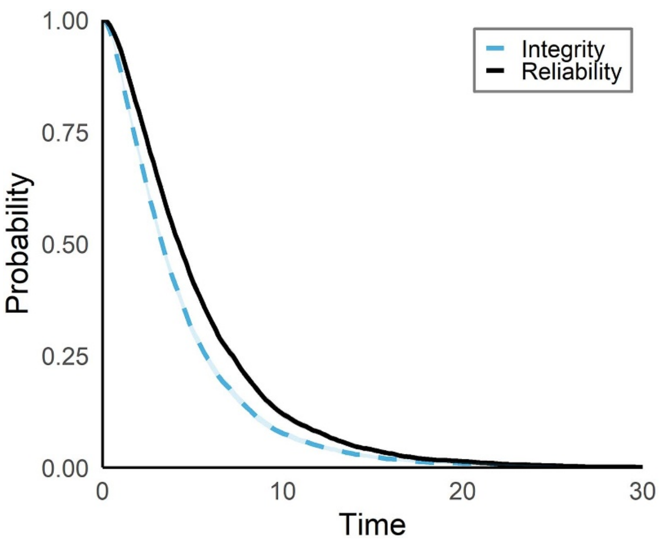

Figure 3. System integrity and reliability curves.

Table 3. System integrity and reliability.

Time	4.83	9.26	13.69	18.12	23.75	28.59	33.83
Integrity	0.583	0.384	0.236	0.154	0.097	0.066	0.042
Reliability	0.975	0.883	0.727	0.543	0.352	0.237	0.141

entire time domain, which is theoretically ensured by the subset relationship established in theorem 1.

To investigate the impact of parameter estimation uncertainty on the system reliability and MTTF, we conducted $N = 1000$ independent simulation replications with $n = 100$ observations each, representing realistic engineering constraints. True reliability metrics were computed from $M = 5000$ Monte Carlo samples as benchmarks.

The procedure comprised the following steps:

- Estimating Kendall's τ and the copula parameter θ from the $n = 100$ sample;
- Generating uniform random vectors $\mathbf{U} = (U_1, \ldots, U_n)$ from the copula $C(\cdot;\theta)$;
- Transforming these into the lifetime space via $X_{ij} = F_i^{-1}(U_{ij})$, where F_i^{-1} denotes the inverse CDF of the ith marginal lifetime;
- Calculating the reliability metrics $R(t)$ and MTTF;
- Constructing 95% bootstrap confidence intervals using $B = 500$ resamples.

Performance was evaluated using bias, RMSE, coverage probability, and relative error across all replications. The results are summarized in tables 4 and 5.

Table 4 reveals that Kendall's τ and copula parameter θ exhibited excellent accuracy with relative bias below 1% and bootstrap confidence interval coverage of 94.0%–95.7%, validating the $B = 500$ bootstrap approach for parameter uncertainty quantification. Table 5 demonstrates that MTTF estimation achieved relative errors within $\pm 2.1\%$, while $R(20)$ showed higher variability with 4.1% positive bias at low correlation ($\tau = 0.29$). Coverage probabilities of 92.4%–93.6% fell slightly below the nominal 95% level, indicating that nonlinear transformations amplify parameter uncertainty beyond what standard bootstrap intervals fully capture.

The results demonstrate that $n = 100$ observations provide sufficient accuracy for reliable parameter estimation (bias $<1\%$), making the proposed approach highly practical for engineering applications where data collection is costly. The bootstrap confidence intervals offer valuable uncertainty bounds for decision-making. For maintenance planning, practitioners should adopt a conservative approach by using the lower bounds of confidence intervals to ensure adequate safety margins, particularly for critical systems where underestimating failure risk is costly.

Table 4. Parameter estimation performance.

Kendall's τ						Copula parameter θ				
True τ	Est. $\hat{\tau}$	Bias	Rel. Bias (%)	RMSE	Coverage (%)	True θ	Est.$\hat{\theta}$	Bias	Rel. Bias (%)	RMSE
0.29	0.292	0.0021	0.7	0.0395	95.7	1.408	1.417	0.0087	0.6	0.0809
0.68	0.680	−0.0003	0.0	0.0286	94.0	3.125	3.147	0.0222	0.7	0.2832
0.81	0.810	−0.0002	0.0	0.0192	94.1	5.263	5.311	0.0482	0.9	0.5356
Average coverage										
τ: 94.6%						θ: 94.6%				

Note: Bootstrap 95% confidence intervals constructed using percentile method with $B = 500$ resamples; True values obtained from $M = 5000$ Monte Carlo samples; Coverage indicates the proportion of times the 95% CI contains the true value. Where Rel. Bias = Relative Bias; Est. = Estimate.

Table 5. System metrics estimation performance at $t = 20$.

Reliability $R(20)$							Mean time to failure (MTTF)				
τ	True $R(20)$	Est. $\hat{R}(20)$	Bias	Rel. Bias (%)	RMSE	Coverage (%)	True MTTF	Est. $\widehat{\text{MTTF}}$	Bias	Rel. Bias (%)	RMSE
0.29	0.458	0.476	0.0186	4.1	0.0529	92.4	21.38	21.82	0.44	2.1	1.37
0.68	0.491	0.492	0.0006	0.1	0.0519	93.5	22.79	22.80	0.01	0.0	1.59
0.81	0.485	0.496	0.0105	2.2	0.0513	93.6	22.83	22.96	0.13	0.6	1.68
Average coverage											
$R(20)$: 93.2%							MTTF: 93.4%				

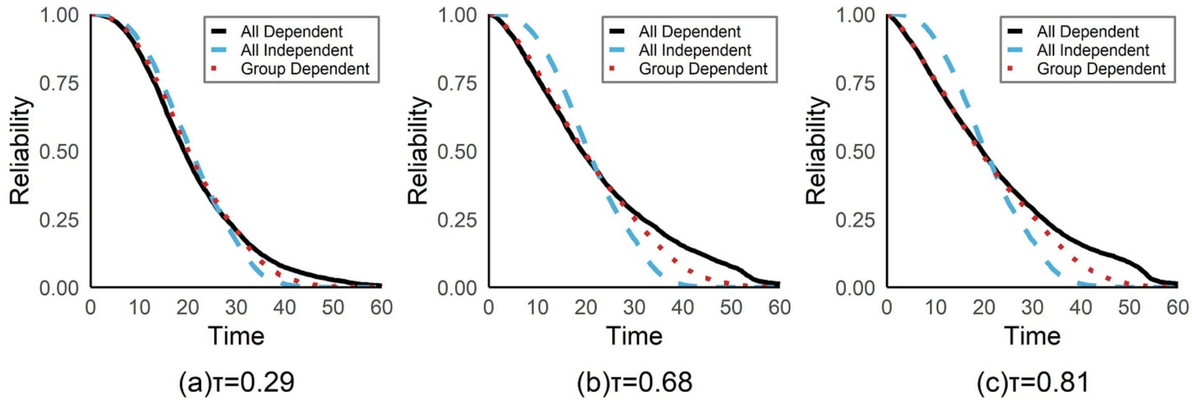

Figure 4. System reliability curves under different component interdependencies.

Experiment 2. The influence of component dependence structures on the $R(t)$ and MTTF.

This experimental study investigates system reliability performance under three distinct component dependence scenarios: the fully dependent case (all components exhibit inter-class and intra-class correlations), the class-independent but intra-class dependent case (components are independent across different categories but correlated within the same category), and the fully independent case (all components are mutually independent).

Figure 4 presents the system reliability curves under different component dependence structures using Gumbel Copula. We can see that as the Kendall's τ increases, the gap between different dependence structures widens, which is consistent with objective reality. In the initial stage, the system reliability $R(t)$ under the three component dependence structures exhibits certain differences: $R(t)$ is higher for the fully independent case. As time progresses, a change point emerges among the $R(t)$ curves at a specific point during system operation (the intersection in figure 4). Beyond this point, the system reliability displays characteristics distinct from those in the initial stage: $R(t)$ is relatively low when components are fully independent, whereas it is relatively high when components are fully dependent.

Therefore, accurately identifying the operating principles of the system is crucial. Based on the assumptions in experiment 2, if dependent components are modeled as independent, the system $R(t)$ is likely to be underestimated. Furthermore, the above results indicate that the correlation among components can extend the system's lifespan to a certain extent.

As shown in table 6, as the Kendall's τ coefficient increases, the MTTF of components under fully dependence and inter-class dependent scenarios gradually decreases. However, changes in Kendall's τ does not impact the MTTF when components are independent.

Table 6. MTTF under different component interdependencies.

τ	Dependent	Dependent (intra), Independent (inter)	Independent
0.29	23.458	22.356	20.756
0.68	22.660	19.069	20.756
0.81	22.019	17.519	20.756

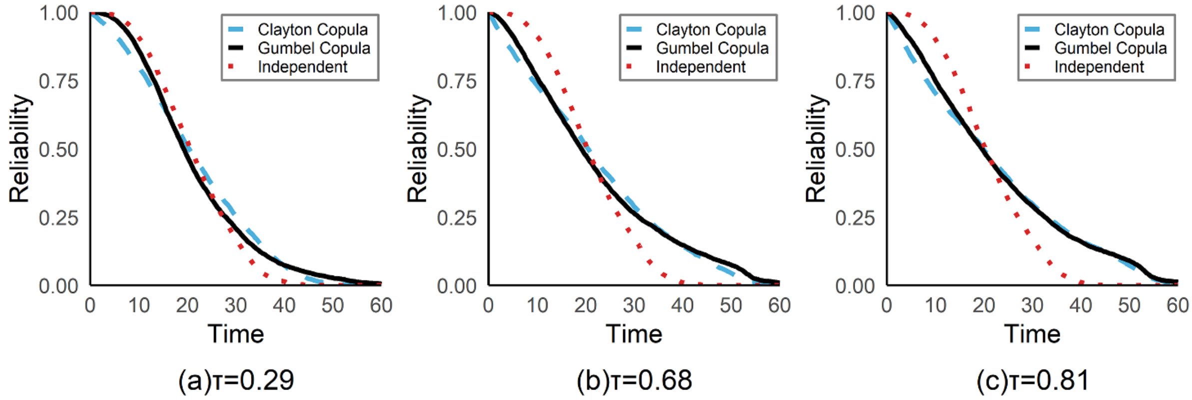

Figure 5. System reliability curves under different Copula-dependent lifetimes.

Experiment 3. The influence of Copula connection functions on the $R(t)$ and MTTF.

This experiment examines the differences in system reliability when the dependence structure between two component types is modeled using different Copula functions, with a focus on comparing the Gumbel Copula and Clayton Copula. The Gumbel Copula is well-suited for scenarios with strong upper-tail dependence, whereas the Clayton Copula is appropriate for cases with strong lower-tail dependence. By simulating reliability curves, we analyze the impact of different Copula models on system reliability assessment, emphasizing the importance of accurately modeling dependence structures.

Figure 5 presents $R(t)$ curves for component lifetimes under three distinct dependence scenarios: Gumbel Copula, Clayton Copula, and independence, with Kendall's τ values of 0.29, 0.68, and 0.81. It shows that when Kendall's τ coefficient is low, the system reliability under Gumbel Copula dependence between component lifetimes is significantly higher than that under Clayton Copula dependence. As Kendall's τ increases, the gap between the system reliability curves for the two copula dependence cases narrows gradually. Thus, when the correlation between component lifetimes is weak, misjudging the dependence structure among component lifetimes will lead to significant errors in system reliability assessments.

The corresponding MTTF values are summarized in table 7, which does not exhibit a clear monotonic trend. As observed in the graph, system reliability is relatively low in the early operational phase but increases in the later phase. This explains the absence of a distinct trend in MTTF. Notably, when component dependence is moderate, the system's MTTF increases.

It should be noted that when reliability curves intersect, relying solely on the MTTF may introduce limitations in assessing system reliability. As the mean value of the reliability function, the MTTF cannot capture the presence of intersection points or reflect the distinct reliability behaviors before and after these critical points under different dependence structures. Therefore, a comprehensive evaluation integrating both reliability curves and MTTF metrics is essential for accurately assessing how dependence structures influence system reliability.

Table 7. MTTF under different Copula-dependent lifetimes.

Kendall's τ	Gumbel Copula	Clayton Copula	Independence
0.29	21.191	21.823	20.756
0.68	23.360	22.536	20.756
0.81	23.283	21.859	20.756

Experiment 4. A comparative study of single-stage versus two-stage degradation processes with weighting considerations.

When component weights are not affected by the system state, the weights exhibit a one-stage degradation characteristic, that is, the parameters remain unchanged throughout the weight degradation process. In the $(w_t-k)_2/n$ system configuration, all parameters strictly adhere to the specifications detailed in table 2, with components exhibiting fully Gumbel Copula dependence ($\tau=0.29$).

Figure 6 presents the $G(t)$ and $R(t)$ curves for both scenarios, providing a comparative visualization of their degradation characteristics. As shown in figure 6, $G(t)$ consistently remains below $R(t)$ at all time points, validating theorem 1. Since $R(t)$ incorporates both normal operating state and degraded state while $G(t)$ captures only normal operating state,

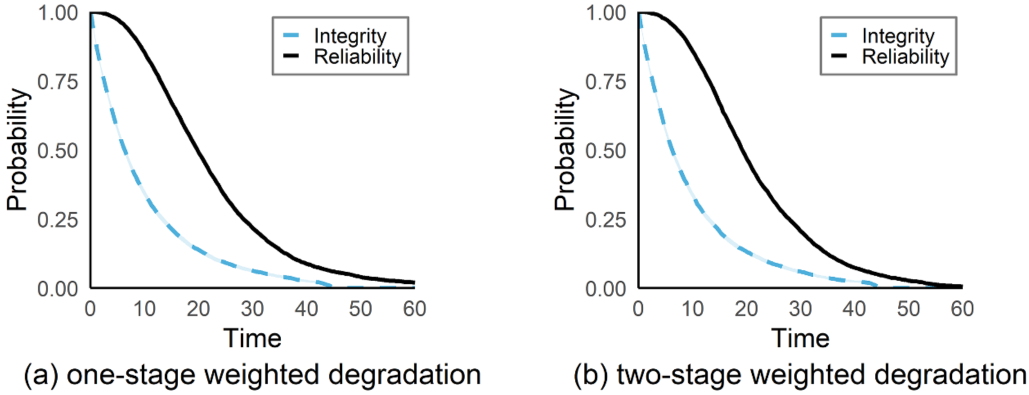

Figure 6. System reliability under different weighted degradation conditions.

Table 8. MTTD, MTTF, and MDOT under different weighted degradation conditions.

Conditions	MTTD	MTTF	MDOT
One-stage weighted degradation	9.589	22.196	12.607
Two-stage weighted degradation	9.589	21.467	11.877

the gap between them represents the probability of degraded operation.

The system integrity curves (blue dotted lines) maintain perfect consistency across both one-stage and two-stage weighted degradation processes. This invariance occurs because, in the two-stage degradation model, the transitions of weight parameters take effect only after the system enters the degraded state. Consequently, they do not affect the critical time at which the system first reaches the degradation threshold, which is intuitively verified in the figure.

The $R(t)$ curve under one-stage weighted degradation is higher than that under two-stage weighted degradation (black solid line). In the two-stage random process, the weighted degradation rate increases significantly after the change point, causing the system to fail faster than in the one-stage scenario, as shown in table 8. The MTTF decreases from 22.196 to 21.467 time units, a reduction of 0.729 time units (3.3%). This acceleration stems from the intensified weight degradation rates in stage 2. Such amplified degradation compresses the degraded operational window: the MDOT contracts from 12.607 to 11.877 time units, confirming that the accelerated post-degradation process leads to earlier system failure without affecting the initial degradation onset time.

Under two-stage weighted degradation, the MTTF is lower, whereas the MTTD remains unchanged in both cases—a finding consistent with the conclusions derived from figure 6.

These findings carry significant ramifications for reliability-centered maintenance strategies. Systems exhibiting two-stage degradation characteristics require heightened inspection frequencies during degraded state operation to preempt catastrophic failures, as the available intervention window is approximately 5.8% shorter than in one-stage systems. The invariance of both $G(t)$ and MTTD (9.589) across degradation models suggests that predictive maintenance algorithms targeting defect onset can be designed independently of subsequent degradation dynamics, simplifying condition monitoring architectures. However, post-defect maintenance planning must account for accelerated deterioration in two-stage systems to avoid premature failures during the compressed MDOT period.

5.2. *Numerical example*

The proposed model exhibits broad applicability across engineering domains where systems comprise heterogeneous components with time-varying capacities and multi-stage degradation behavior. To demonstrate this versatility, we present two representative application scenarios with varying levels of parametric detail.

First, as a conceptual illustration of parameter interpretation, consider a grid-scale battery energy storage array integrating lithium iron phosphate cells (Type A), nickel–manganese–cobalt cells (Type B), and lithium titanate cells (Type C). Weights represent discharge capacity (Ah) that degrades with cycle aging, exhibiting accelerated decline once state of health falls below 80%, thereby aligning with the two-stage degradation mechanism in our model. Cells within thermal zones demonstrate correlated aging due to shared temperature profiles, captured by Clayton copula. Critical parameters derive from: (i) periodic capacity tests for $W_i(t)$; (ii) accelerated aging experiments establishing degradation rates α_i, β_i; (iii) multi-cell monitoring data for correlation structure τ.

Building upon this conceptual framework, we now proceed to a comprehensive validation case with complete parameter specifications. Specifically, we apply the model to a pneumatic energy system from an advanced fighter aircraft, employing numerical simulation analysis to provide rigorous computational verification. This case study systematically evaluates system performance metrics such as reliability, thereby demonstrating the practical value of the theoretical framework. The system employs four on-board air

Table 9. Simulation parameters.

Parameter name	Type A	Type B	Type C
Number	$n_1 = 2$	$n_2 = 1$	$n_3 = 1$
Distribution	Exp(0.1)	Exp(0.2)	Exp(0.3)
Initial weight	$W_1(0) = 1$	$W_2(0) = 2$	$W_3(0) = 3$
Stage-1 weighted loss function	$\mathrm{Ga}(0.05t, 0.1)$	$\mathrm{Ga}(0.04t, 0.1)$	$\mathrm{Ga}(0.03t, 0.1)$
Stage-2 weighted loss function	$\mathrm{Ga}(0.09t, 0.1)$	$\mathrm{Ga}(0.08t, 0.1)$	$\mathrm{Ga}(0.07t, 0.1)$
System state	Normal	Degraded	
Threshold symbol	K_1 W_1	K_2 W_2	
Threshold value	2 5	1 3	

pressurization units for inflation and replenishment, each with distinct output power characteristics. The overall pressurization subsystem can be regarded as a weighted system, where the collective output performance depends on the coordinated operating states of these units [45]. In this configuration, the system comprises three types of components, whose lifetimes are linked by a Gumbel Copula with $\tau = 0.29$. The specific parameters are presented in table 9 below.

The system $G(t)$ and $R(t)$ respectively are expressed as follow:

$$G(t) = \sum_{a_1=0}^{2} \sum_{\substack{a_2=0 \\ a_1+a_2+a_3 \geqslant 2}}^{1} \sum_{a_3=0}^{1} \frac{\gamma\left(0.17t, 0.1\left(\sum_{i=1}^{3} a_i W_i(0) - 5\right)\right)}{\Gamma(0.17t)} \times \prod_{i=1}^{3} \binom{n_i}{a_i} \left(\sum_{k_1=0}^{a_1} \sum_{k_2=0}^{a_2} \sum_{k_3=0}^{a_3} (-1)^{k_1+k_2+k_3} \prod_{i=1}^{3} \binom{a_i}{k_i} A_{1:3}(t) \right) \tag{23}$$

$$R(t) = \int_0^t \left[\sum_{b_1=0}^{2} \sum_{\substack{b_2=0 \\ b_1+b_2+b_3 \geqslant 1}}^{1} \sum_{b_3=0}^{1} \frac{\gamma\left(0.17t, 0.1\left(\sum_{i=1}^{3} a_i W_i(0) - 5\right)\right)}{\Gamma(0.17t)} \times \prod_{i=1}^{3} \binom{n_i}{b_i} \left(\sum_{k_1=0}^{b_1} \sum_{k_2=0}^{b_2} \cdots \sum_{k_3=0}^{b_3} (-1)^{k_1+k_2+k_3} \prod_{i=1}^{3} \binom{b_i}{k_i} B_{1:3}(t) \right) \right] f_{T_1}(z)\,\mathrm{d}z. \tag{24}$$

The simulation results for system $G(t)$ and $R(t)$ are presented in figure 7 below.

Consistent with the theoretical prediction from theorem 1, the integrity curve lies strictly below the reliability curve. The observation that $G(t) < R(t)$ for all $t > 0$ validates our theoretical framework. This behavior reflects the dynamic transition of the system through different operational phases: from normal operation, through the degraded state, and eventually to complete failure.

The MTTF of the system was approximated via Monte Carlo simulation, where the average of simulated operation times was used as the estimated MTTF value. When the number of simulations is sufficiently large, this estimate converges to the true MTTF. Based on 6000 simulation runs, the estimated MTTF was determined to be 7.0288.

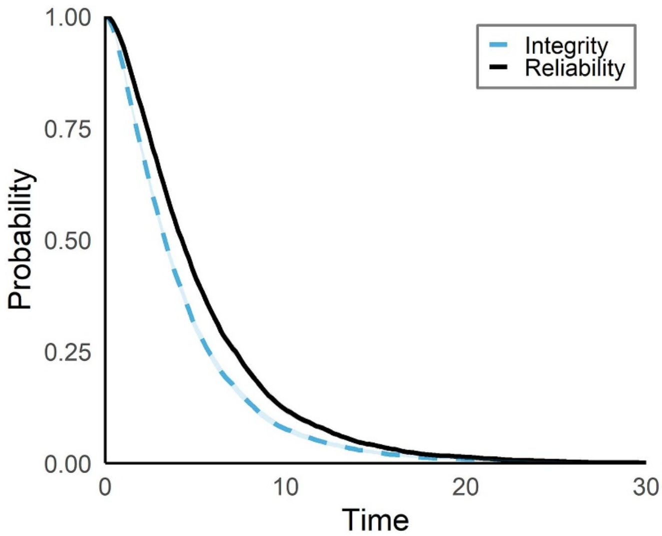

Figure 7. System integrity and reliability curves.

6. Conclusion

This study proposes a novel weighted k/n system incorporating multi-type components with continuous random weights and two-stage degradation behavior. The main contributions and findings are summarized as follows:

- The two-stage continuous random weight degradation model accurately captures component performance decline, with degraded state operation reducing MTTF by 3.3% compared to one-stage weighted degradation while leaving MTTD unchanged.
- Misspecification of component dependence structures leads to significant estimation errors when dependent components are incorrectly modeled as independent. Component dependence can enhance system reliability in later operational phases, despite accelerating early failures.
- When correlation is weak ($\tau = 0.29$), incorrectly choosing between Gumbel and Clayton copulas causes significant reliability estimation errors. As correlation strengthens

($\tau \to 0.81$), differences between copula models diminish, indicating dependence structure identification is most critical for weakly correlated systems.

- With $n = 100$ observations, parameter estimation achieves relative bias $< 1\%$ and 94% confidence interval coverage. However, MTTF estimation at low correlation ($\tau = 0.29$) shows 4.1% positive bias, suggesting conservative maintenance planning should use lower confidence bounds for critical systems.
- The inequality $G(t) \leqslant R(t)$ holds throughout system operation, confirming that simultaneous consideration of component count and total weight provides refined state discrimination for maintenance decision-making.

Several promising avenues for future research include:

- Extending the model to multi-stage degradation processes with multiple intermediate states, and developing dynamic weight optimization and maintenance strategies;
- Incorporating time-varying copula structures to capture evolving dependencies, and enhancing computational efficiency through approximation methods;
- Validating with extensive real-world industrial data, and integrating machine learning for data-driven model refinement and adaptive predictions.

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Appendix A. Proof: theorem 1

Define the events:

$$
\begin{aligned}
A_1(t) &= \{K(t) \geqslant K_1\} \\
A_2(t) &= \{W(t) \geqslant W_1\} \\
B_1(t) &= \{K(t) \geqslant K_2\} \\
B_2(t) &= \{W(t) \geqslant W_2\}.
\end{aligned}
$$

Then:

$$
\begin{aligned}
G(t) &= P(A_1(t) \cap A_2(t)) \\
R(t) &= P(B_1(t) \cap B_2(t)).
\end{aligned}
$$

Since $K_2 < K_1$ and the number of surviving components $K(t)$ is monotonically non-increasing in t, we have $A_1(t) \subseteq B_1(t)$.

Similarly, since $W_2 < W_1$ and the total weight $W(t)$ is monotonically non-increasing in t (by model assumption (I,c)), we have $A_2(t) \subseteq B_2(t)$.

Therefore:

$$A_1(t) \cap A_2(t) \subseteq B_1(t) \cap B_2(t).$$

By the monotonicity of probability:

$$P(A_1(t) \cap A_2(t)) \leqslant P(B_1(t) \cap B_2(t)).$$

Hence $G(t) \leqslant R(t)$ for all $t \geqslant 0$.

Appendix B. Proof: property 1

$$
\begin{aligned}
G(t) &= P\{T_1 > t\} \\
&= P\{T_{1a} > t, T_{1b} > t\} \\
&= P\left\{\sum_{i=1}^{m}\sum_{j=1}^{n_i} I_{ij}(t) \geqslant K_1, \sum_{i=1}^{m}\sum_{j=1}^{n_i} W_{ij}^{(1)}(t) I_{ij}(t) \geqslant W_1\right\}.
\end{aligned} \tag{25}
$$

Based on the conditional probability formula and the total probability theorem, we have that

$$
\begin{aligned}
&P\left\{\sum_{i=1}^{m}\sum_{j=1}^{n_i} I_{ij}(t) \geqslant K_1, \sum_{i=1}^{m}\sum_{j=1}^{n_i} W_{ij}^{(1)}(t) I_{ij}(t) \geqslant W_1\right\} \\
&= P\left\{\sum_{i=1}^{m}\sum_{j=1}^{n_i} I_{ij}(t) \geqslant K_1\right\} \\
&\times P\left\{\sum_{i=1}^{m}\sum_{j=1}^{n_i} W_{ij}^{(1)}(t) I_{ij}(t) \geqslant W_1 \middle| \sum_{i=1}^{m}\sum_{j=1}^{n_i} I_{ij}(t) \geqslant K_1\right\} \\
&= \sum_{a_1=0}^{n_1}\sum_{a_2=0}^{n_2}\cdots\sum_{a_m=0}^{n_m} P\left\{\sum_{j=1}^{n_1} I_{1j}(t) = a_1, \sum_{j=1}^{n_2} I_{2j}(t) = a_2, \ldots, \sum_{j=1}^{n_m} I_{mj}(t) = a_m\right\} \\
&\times P\left\{\sum_{i=1}^{m} a_i \geqslant K_1 \middle| \sum_{j=1}^{n_1} I_{1j}(t) = a_1, \sum_{j=1}^{n_2} I_{2j}(t) = a_2, \ldots, \sum_{j=1}^{n_m} I_{mj}(t) = a_m\right\} \\
&\times P\left\{\sum_{i=1}^{m}\sum_{j=1}^{n_i} W_{ij}^{(1)}(t) I_{ij}(t) \geqslant W_1 \middle| \sum_{i=1}^{m} a_i \geqslant K_1\right\}
\end{aligned} \tag{26}
$$

where

$$
\begin{aligned}
&P\left\{\sum_{j=1}^{n_1} I_{1j}(t) = a_1, \sum_{j=1}^{n_2} I_{2j}(t) = a_2, \ldots, \sum_{j=1}^{n_m} I_{mj}(t) = a_m\right\} \\
&= P\left\{\sum_{j=1}^{n_1} (1 - I_{1j}(t)) = n_1 - a_1, \cdots, \sum_{j=1}^{n_m} (1 - I_{mj}(t)) = n_m - a_m\right\} \\
&= \prod_{i=1}^{m} \binom{n_i}{a_i} \left(\sum_{k_1=0}^{a_1}\sum_{k_2=0}^{a_2}\cdots\sum_{k_m=0}^{a_m} (-1)^{k_1+k_2+\cdots+k_m} \prod_{i=1}^{m} \binom{a_i}{k_i} A_{1:m}(t)\right).
\end{aligned} \tag{27}
$$

Note that in formula (26), the conditional probability $P\{\sum_{i=1}^{m} a_i \geqslant K_1 | \sum_{j=1}^{n_1} I_{1j}(t) = a_1, \sum_{j=1}^{n_2} I_{2j}(t) = a_2, \ldots, \sum_{j=1}^{n_m} I_{mj}(t) = a_m\}$ can only take binary values 1 or 0. Substituting formula (27) into (26) for simplification to obtain the system integrity function (9).

Appendix C. Proof: property 2

$$
\begin{aligned}
R(t) &= P\{T_2 > t\} \\
&= P\{T_1 + \Delta T_{2a} > t, T_1 + \Delta T_{2b} > t\}.
\end{aligned} \tag{28}
$$

The duration ΔT_{2a} of the degraded state during which the system eventually fails because the number of surviving components falls below K_2, the duration ΔT_{2b} of the degraded state during which the system eventually fails because the total weight of surviving components falls below W_2. Given the system normal operating time T_1 and its probability density function $f_{T_1}(z)$, the reliability function $R(t)$ can be expressed as

$$\begin{aligned} R(t) &= \int_0^t P\{T_1+\Delta T_{2a}>t, T_1+\Delta T_{2b}>t|T_1=z\} f_{T_1}(z)\,\mathrm{d}z \\ &= \int_0^t P\{z+\Delta T_{2a}>t, z+\Delta T_{2b}>t\} f_{T_1}(z)\,\mathrm{d}z \\ &= \int_0^t P\left\{\sum_{i=1}^m\sum_{j=1}^{n_i} I_{ij}(t)\geqslant K_2, \sum_{i=1}^m\sum_{j=1}^{n_i} W_{ij}^{(2)}(t|z)\, I_{ij}(t)\geqslant W_2\right\} f_{T_1}(z)\,\mathrm{d}z. \end{aligned} \tag{29}$$

According to the law of total probability, the expression within the integral sign can be further calculated as follows:

$$\begin{aligned} &P\left\{\sum_{i=1}^m\sum_{j=1}^{n_i} I_{ij}(t)\geqslant K_2, \sum_{i=1}^m\sum_{j=1}^{n_i} W_{ij}^{(2)}(t|z)\, I_{ij}(t)\geqslant W_2\right\} \\ &= P\left\{\sum_{i=1}^m\sum_{j=1}^{n_i} I_{ij}(t)\geqslant K_2\right\}\times P\left\{\sum_{i=1}^m\sum_{j=1}^{n_i} W_{ij}^{(2)}(t|z)\geqslant W_2 \middle| \sum_{i=1}^m\sum_{j=1}^{n_i} I_{ij}(t)\geqslant K_2\right\} \\ &= \sum_{b_1=0}^{n_1}\sum_{b_2=0}^{n_2}\cdots\sum_{b_m=0}^{n_m} P\left\{\sum_{j=1}^{n_1} I_{1j}(t)=b_1, \sum_{j=1}^{n_2} I_{2j}(t)=b_2,\cdots,\sum_{j=1}^{n_m} I_{mj}(t)=b_m\right\} \\ &\times P\left\{\sum_{i=1}^m b_i\geqslant K_2 \middle| \sum_{j=1}^{n_1} I_{1j}(t)=b_1, \sum_{j=1}^{n_2} I_{2j}(t)=b_2,\ldots,\sum_{j=1}^{n_m} I_{mj}(t)=b_m\right\} \\ &\times P\left\{\sum_{i=1}^m\sum_{j=1}^{b_i} W_{ij}^{(2)}(t|z)\geqslant W_2 \middle| \sum_{i=1}^m b_i\geqslant K_2\right\}. \end{aligned} \tag{30}$$

At time t, the joint probability of observing exactly $b_1,\cdots,b_m$ surviving components in categories $1,\cdots,m$, respectively, is given by

$$\begin{aligned} &P\left\{\sum_{j=1}^{n_1} I_{1j}(t)=b_1, \sum_{j=1}^{n_2} I_{2j}(t)=b_2,\cdots,\sum_{j=1}^{n_m} I_{mj}(t)=b_m\right\} \\ &= \prod_{i=1}^m\binom{n_i}{b_i}\left(\sum_{k_1=0}^{b_1}\sum_{k_2=0}^{b_2}\cdots\sum_{k_m=0}^{b_m}(-1)^{k_1+k_2+\cdots+k_m}\prod_{i=1}^m\binom{b_i}{k_i}B_{1:m}(t)\right). \end{aligned} \tag{31}$$

By substituting formulas (30) and (31) into formula (29), property 2 can be derived.

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